The behaviour of a turbulent boundary layer near separation

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Stratford's method of computing the development of a turbulent boundary layer in a strong adverse pressure gradient, based on division of the layer into an inner equilibrium layer and an outer layer of almost constant total-head, is simple and is based on current knowledge of the properties of turbulent flows, but the correspondence with experiment is worse than would be expected. The causes of this are investigated, first by testing the basic assumptions against the measurements of Schubauer & Klebanoff and then by calculating from the theory properties of the layer at intermediate stages of the development towards separation. It is shown that the theoretical condition for zero wall stress (a relation between pressure rise and pressure gradient) is always satisfied twice, first at an intermediate stage of development with wall stress about one-fifth of the initial value and second at the real position of zero stress. This behaviour involves a rapid decrease of pressure gradient in the neighbourhood of separation, caused by rapid thickening of the layer, and it is shown that the pressure distribution near separation depends only on the pressure rise to separation and the characteristics of the initial boundary layer and not on the geometry of the flow. With more measurements of boundary layers separating in strong adverse pressure gradients, these characteristic distributions could be determined as a one-parameter family. Their use for the prediction of position and pressure rise to separation in flows with specified boundaries without prior knowledge of the pressure distribution is discussed.

1. Introduction

The development of a turbulent boundary layer in an adverse pressure gradient is a problem that has received much attention, but most attempts have started with the equation for the overall momentum balance in the layer and then proceed to make empirical assumptions about the form of the velocity distributions. In recent years these assumptions have become more realistic, particularly by using the 'law of the wall' for the inner layer, and the ability to predict the position of separation in a specified pressure distribution is good, but the assumptions are not approximations that become exact in relevant limiting conditions. A theory satisfying this last requirement has been proposed by Stratford (1959) who points out that the flow close to the wall is determined by the local stress distribution and is otherwise independent of the past history of the layer, while the outer part of the flow develops nearly independently of the

Reynolds stresses if the adverse pressure gradient is large. If the whole layer at any stage of the development is adequately described by combining an inner layer of structure determined by the local stress distribution with an outer layer on which Reynolds stresses have had no appreciable effect, comparatively simple equations can be obtained giving the layer characteristics as functions of the pressure rise, the local pressure gradient and the initial characteristics of the layer. In particular, they lead to a criterion for zero wall stress in the form of a relation between pressure rise and pressure gradient that must be satisfied. Although some details of the original formulation need amendment (Townsend 1960), the basic assumptions of the theory are so plausible and in keeping with current knowledge of the properties of turbulent flows that it is surprising to find comparatively poor agreement between the observed and predicted pressure rises at flow separation. For example, in the separating boundary layer studied by Schubauer & Klebanoff (1951), the zero-stress criterion is satisfied for a pressure recovery coefficient of 0·38† but the layer does not separate until a coefficient of 0·51 is reached.

In this paper, some experimental evidence for the validity of the basic assumptions of the theory is described with the object of justifying its description of intermediate stages of the development towards zero stress and flow separation. From the theoretical analysis it appears that the zero-stress criterion is always satisfied for two values of the pressure rise, first at an intermediate stage with finite wall stress and then at the real separation point. This phenomenon requires a considerable weakening of the adverse pressure gradient as the layer approaches separation, and the implications are discussed in relation to the problem of flow separation in a system with given boundaries but undetermined distribution of pressure.

2. Basic assumptions

We consider a turbulent boundary layer on a plane surface with two-dimensional mean flow and developing in a known adverse pressure gradient. The flow is described in the usual co-ordinates with Ox along the surface and in the direction of mean flow and Oy normal to the surface. At the position $x = x_0$, the pressure‡ is a minimum P_0 and, for larger values of x, it increases rapidly until the layer separates. At the pressure minimum, the equilibrium layer is nearly one of constant stress and the velocity distribution over the inner fifth of the total layer thickness is described by the logarithmic 'law of the wall',

$$\begin{aligned} U &= \frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\frac{\tau_0^{\frac{1}{2}} y}{\nu} + A\right] \\ &= U_0(1+\gamma\log\eta), \end{aligned} \tag{2.1}$$

† Not 0·42 as in an earlier paper (Townsend 1960) which used $\gamma = 0{\cdot}075$ instead of the correct value, 0·083 (see § 4). Stratford (1959) gives $C_{p,s} = 0{\cdot}44$, but his form of the theory uses different junction conditions.

‡ Pressures and stresses are 'kinematic', i.e. the mechanical values divided by the fluid density.

where U_0 is the free-stream velocity at $x = x_0$,
τ_0 is the wall stress at $x = x_0$,
$\gamma = \tau_0^{\frac{1}{2}}/(KU_0)$,
$\eta = y/\delta_0$,
$\delta_0 = \nu/\tau_0^{\frac{1}{2}} \exp(\gamma^{-1} - A)$,
ν is the kinematic viscosity,
K is the Karman constant ($= 0{\cdot}41$),
A is an additive constant ($= 2{\cdot}3$).

If the adverse pressure gradient is large compared with the stress gradients in the layer at the pressure minimum, the rates of energy production and dissipation in the outer parts of the flow are too slow to modify appreciably the original Reynolds stresses and these will be nearly unchanged along streamlines. Then the gradients of Reynolds stresses will be diminished by the flow expansion and they can cause no appreciable changes in total-head along streamlines in the outer flow.† In symbols, if

$$dP/dx \gg \tau_0/\delta_0,$$

$$P + \tfrac{1}{2}U^2 \equiv P_0 + \tfrac{1}{2}U_0^2 + \tfrac{1}{2}(U^2 - U_1^2) = f(\psi), \tag{2.2}$$

and

$$\tau = g(\psi), \tag{2.3}$$

where P is the local pressure,
U_1 is the local free-stream velocity,
$\psi = \int_0^y U_1(y')\,dy'$ is the stream function.

In the inner part of the layer the rates of production and dissipation of turbulent energy are large enough to modify considerably the Reynolds stresses, and neither Reynolds stress nor total head is expected to remain constant along streamlines; but if these rates are also large compared with the rate of energy gain by advection, the inner layer is nearly an equilibrium layer (Townsend 1961). Then the velocity distribution is determined by the local distribution of stress through the relation

$$\frac{\partial U}{\partial y} = \frac{\tau^{\frac{1}{2}}}{Ky}\left(1 - B\frac{y}{\tau}\left|\frac{\partial \tau}{\partial y}\right|\right), \tag{2.4}$$

where τ is the local shear stress, and B is an absolute constant ($= 0{\cdot}18$).

The whole layer is now considered to be composed of two distinct but adjacent parts, an inner equilibrium layer and an outer layer with constant total-head and Reynolds stress along streamlines, the position of the junction being determined by requiring continuity of mean velocity and of Reynolds stress (Townsend 1960). Assuming the stress distribution in the equilibrium layer to be nearly linear, i.e.

$$\tau = \tau_1 + \alpha y, \tag{2.5}$$

it becomes possible to calculate the boundary-layer characteristics as functions of the pressure coefficient, $C_p = (P - P_0)/\frac{1}{2}U_0^2$, dC_p/dx, and the parameters describing the layer at the pressure minimum.

† Stratford (1959) made an allowance for change of head by the original Reynolds stresses.

Reasons for believing that this procedure should give an approximate description of the behaviour of the layer have been given before in some detail (Stratford 1959; Townsend 1960), but it is instructive to compare the assumed behaviour with the behaviour of the turbulent layer studied in detail by Schubauer & Klebanoff (1951). In this layer, the effective Reynolds number at the pressure

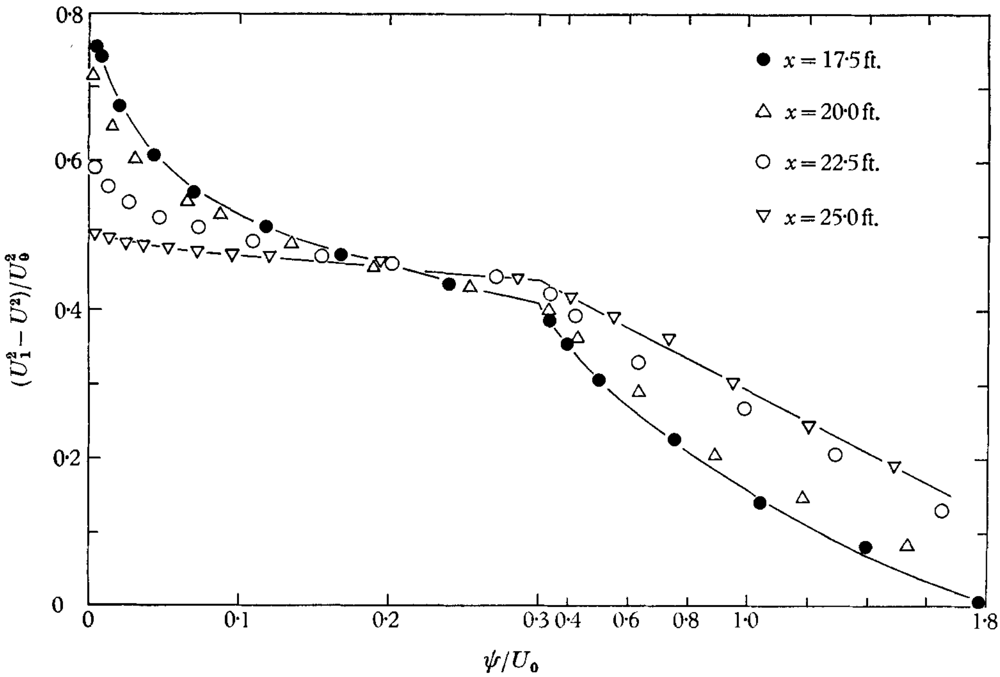

FIGURE 1. Variation of total-head along streamlines in a separating boundary layer (Schubauer & Klebanoff 1951). (N.B. change of horizontal scale at $\psi/U_0 = 0{\cdot}3$ in.)

minimum was $14{\cdot}3 \times 10^6$ and the ratio $(\tau_0/\delta_0)/(dP/dx)$ about 0·07, indicating a sufficiently strong pressure gradient for application of the theory. For our present purposes, the essential assumptions are:

(i) Total head is conserved between x_0 and x on the streamline passing through the junction of the two layers at the position x.

(ii) Reynolds stress is constant along the same streamline.

(iii) The stress distribution in the inner layer is nearly linear.

(iv) The velocity distribution in the inner layer can be found by integration of equation (2.4).

(v) Mean velocity and Reynolds stress are continuous across the junction of the two layers.

In figure 1, the variation of total head along streamlines is displayed by plotting $(U_1^2 - U^2)/U_0^2$ against $U_0^{-1}\psi$ at different stages of development towards separation. As would be expected, gains of total head occur in the equilibrium layer for $U_0^{-1}\psi$ less than 0·1 in. and there are appreciable losses in the outermost layers, but the changes are very small in the range, 0·15 to 0·3 in, where the junction points occur. It may appear singular that total head is conserved most accurately along streamlines through junction points, but in fact these points in the theoretical flow correspond to stress maxima in the real flow and so, locally, are positions where the rate of change of total head is zero.

The degree of conservation of Reynolds stress is shown in figure 2 by plotting the stress coefficient,† $C_\tau = \tau/\tfrac{1}{2}U_0^2$, against $U_0^{-1}\psi$. The same features appear, strongly modified distributions near the wall and nearly unchanged distributions

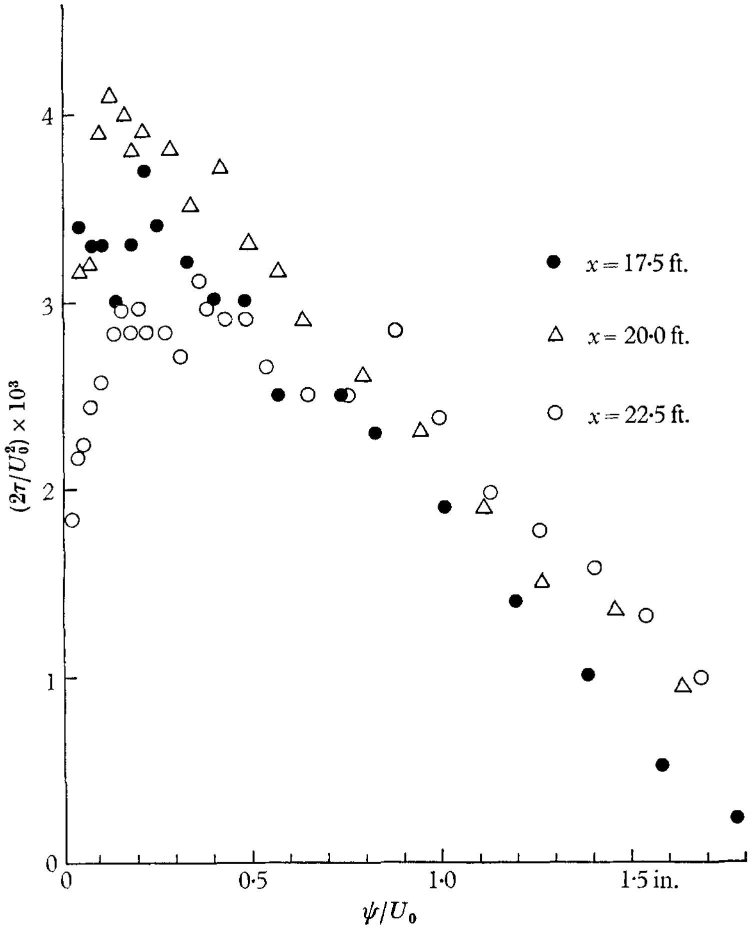

FIGURE 2. Variation of Reynolds stress along streamlines in a separating boundary layer (Schubauer & Klebanoff 1951).

in the outer parts of the flow. Linearity of the stress distribution may be confirmed from these same measurements (figure 3) or indirectly by comparing the observed velocity distributions with the forms found by substituting the linear stress distribution (2.5) in equation (2.4). For a smooth surface with $\tau_1^{\frac{3}{2}}/\alpha\nu > 20$, it is

$$U = \frac{\tau_1^{\frac{1}{2}}}{K}\left[\log\left(\frac{4\tau_1^{\frac{3}{2}}}{\alpha\nu}\,\frac{(\tau_1+\alpha y)^{\frac{1}{2}}-\tau_1^{\frac{1}{2}}}{(\tau_1+\alpha y)^{\frac{1}{2}}+\tau_1^{\frac{1}{2}}}\right)+A-2(1-B)\right]+\frac{2(1-B)}{K}(\tau_0+\alpha y)^{\frac{1}{2}} \quad (2.6)$$

† The measured values of Schubauer & Klebanoff are about 50 % greater than the real ones, but relative values seem trustworthy except close to separation.

and this distribution, first obtained by Szablewski (1954) for $B = 0$, has been verified in some detail by Szablewski (1960) and, for the special case $\alpha y \gg \tau_1$, by Townsend (1961).

Validity of the junction conditions—continuity of mean velocity and Reynolds stress—is more difficult to establish from observations so that comparison of

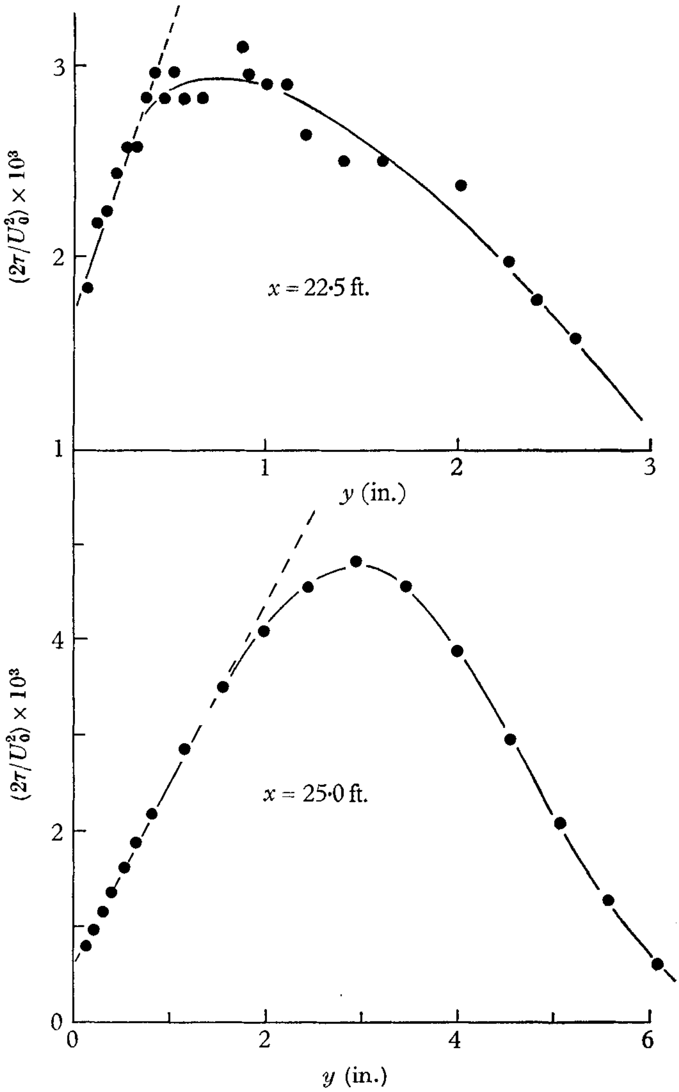

FIGURE 3. Distributions of Reynolds stress in a separating boundary layer (Schubauer & Klebanoff 1951).

observation with deductions from the theory provides the best confirmation. In his original paper, Stratford used continuity of mean velocity and mean velocity gradient as junction conditions, but velocity gradient decreases with distance from the wall in both layers so that continuity of gradient at the junction prevents the appearance of the characteristic point of inflexion in the velocity profile.

Examination of the profiles shows clearly the necessity for a considerable discontinuity in gradient between the distributions appropriate to the two layers (figure 4).

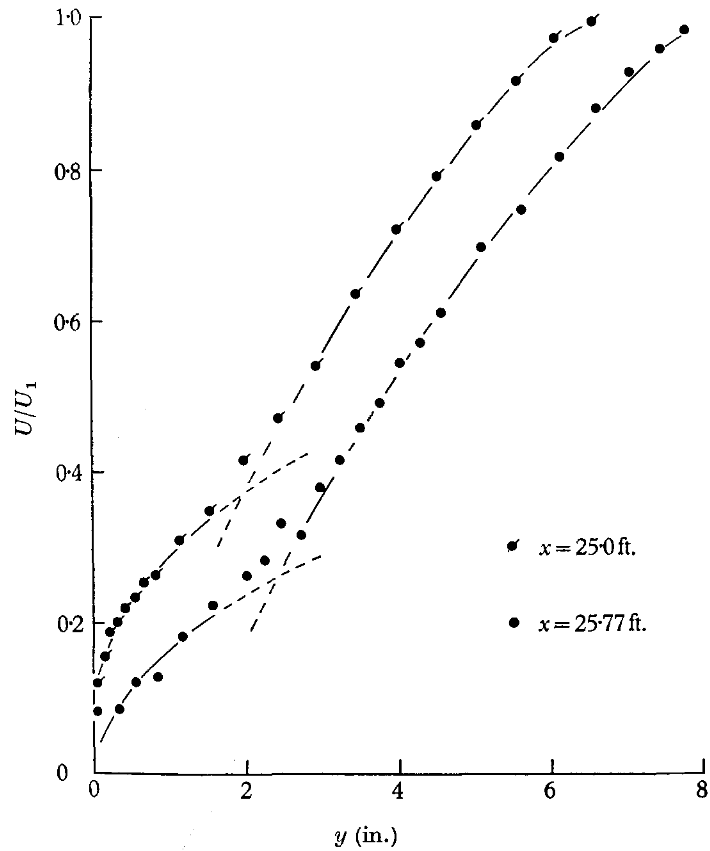

FIGURE 4. Distributions of mean velocity in a separating boundary layer (Schubauer & Klebanoff 1951), showing inner (parabolic) and outer distributions.

3. Development of the boundary layer

In a strong adverse pressure gradient, the stress at the wall decreases steadily from its initial value, τ_0, to zero at the position of flow separation, and it is natural, as well as convenient, to seek a condition that the wall stress should have fallen to a fraction, t^2, of its initial value, i.e.

$$\tau_1 = t^2 \tau_0.$$

At this section of the flow, the junction between the two layers is at (x, y_s) where the conditions of stress continuity and conservation of stress along the streamline through (x, y_s) require that

$$\left.\begin{aligned} \tau_1 + \alpha y_s &= \tau_0, \\ \alpha y_s &= (1-t^2)\tau_0 \end{aligned}\right\} \tag{3.1}$$

(this streamline is assumed to lie within the constant stress layer at $x = x_0$). At the junction, from equation (2.6) and using this result,

$$U_s = t\frac{\tau_0^{\frac{1}{2}}}{K}\left[\log\left(\frac{4t^3(1-t)}{1+t}\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}\right)+A+2\left(\frac{1}{t}-1\right)(1-B)\right], \tag{3.2}$$

$$\psi_s = t(1-t^2)\frac{\tau_0^{\frac{3}{2}}}{\alpha K}\left[\log\left(\frac{4t^3(1-t)}{1+t}\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}\right)+A+\frac{2}{3}\frac{2-t-4t^2}{t(1+t)}(1-B)-\frac{2t}{1+t}B\right]. \tag{3.3}$$

The streamline through (x, y_s) passes through $(x_0, \eta_0\delta_0)$ if

$$\psi_s = U_0\eta_0\delta_0(1-\gamma+\gamma\log\eta_0) \tag{3.4}$$

and the condition of constant total-head along this streamline is that

$$P_0+\tfrac{1}{2}U_0^2(1+\gamma\log\eta_0)^2 = P+\tfrac{1}{2}U_s^2,$$

i.e.

$$C_p = \frac{P-P_0}{\frac{1}{2}U_0^2} = \gamma^2\left[\left(\log\eta_0+\frac{1}{\gamma}\right)^2 - t^2\left(\log\left(\frac{4t^3(1-t)}{1+t}\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}\right)+A+2\left(\frac{1}{t}-1\right)(1-B)\right)^2\right]. \tag{3.5}$$

The equations (3.4) and (3.5) determine the wall stress as a function of the pressure rise and the stress gradient, α.

First consider the solution of these equations for not too small values of the stress ratio t^2. Equation (3.4) may be put in the form

$$\eta_0(\log\eta_0+\gamma^{-1}-1) = \frac{t(1-t^2)}{K\gamma}\frac{\nu}{U_0\delta_0}\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu} \times\left[\log\left(\frac{4t^3(1-t)}{1+t}\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}\right)+A+\frac{2}{3}\frac{2-t-4t^2}{t(1+t)}(1-B)-\frac{2t}{1+t}B\right] \tag{3.6}$$

and an approximate solution is

$$\log\eta_0 = \log\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}+\log t(1-t^2)-\log\left(K\gamma\frac{U_0\delta_0}{\nu}\right),$$

valid for large values of $\log(\tau_0^{\frac{3}{2}}/\alpha\nu)$. A closer approximation can be obtained by iteration. It is

$$\log\eta_0 = \log\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}+\log t(1-t^2)-\log\left(K\gamma\frac{U_0\delta_0}{\nu}\right)+\frac{F(t)}{\log\tau_0^{\frac{3}{2}}/(\alpha\nu)}, \tag{3.7}$$

where $$F(t) = 1+\log\frac{4t^2}{(1+t)^2}+\frac{2}{3}\frac{2-t-4t^2}{t(1+t)}(1-B)-\frac{2t}{1+t}B$$

and the approximation is a good one if

$$F(t) \ll \log\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}. \tag{3.8}$$

If $\log\tau_0^{\frac{3}{2}}/\alpha\nu$ is more than four, this restricts t to the range 0·2 to 1·0. Substituting (3.7) in equation (3.5) and using the relation $\delta_0 = \nu\tau_0^{-\frac{1}{2}}\exp(\gamma^{-1}-A)$,

$$C_p/\gamma^2 = \left(\log\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}+A+\log t(1-t^2)+\frac{F(t)}{\log(\tau_0^{\frac{3}{2}}/\alpha\nu)}\right)^2 - t^2\left(\log\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}+A+2\left(\frac{1}{t}-1\right)(1-B)+\log\frac{4t^3(1-t)}{1+t}\right)^2, \tag{3.9}$$

a relation between C_p/γ^2, $\log(\tau_0^{\frac{3}{2}}/\alpha\nu)$ and the stress ratio t^2, approximately true if t exceeds 0·2.

For very small values of t,† the two equations become

$$\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}\left[t\left\{\log\left(4t^3\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}\right)+A\right\}+\tfrac{4}{3}(1-B)\right] = \eta_0 e^{\gamma^{-1}-A}\left[\log\eta_0+\gamma^{-1}-1\right] \quad (3.10)$$

and
$$C_p/\gamma^2 = \left(\log\eta_0+\frac{1}{\gamma}\right)^2-\left(t\left\{\log\left(4t^3\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}\right)+A\right\}+2(1-B)\right)^2. \quad (3.11)$$

The second term in (3.11) is fairly small compared with the first and can be replaced by $4(1-B)^2$ without serious error if t is less than 0·1, and then

$$\log\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu} = \frac{\left(\dfrac{C_p}{\gamma^2}+4(1-B)^2\right)^{\frac{1}{2}}+\log\left[\left(\dfrac{C_p}{\gamma^2}+4(1-B)^2\right)^{\frac{1}{2}}-1\right]-A-\log\tfrac{4}{3}(1-B)-\dfrac{\log 4t^3+A}{\frac{4}{3}(1-B)}t}{1+\dfrac{3}{4}\dfrac{t}{1-B}}. \quad (3.12)$$

At the position of zero wall stress, $t=0$ and there

$$\log\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu} = \left(\frac{C_p}{\gamma^2}+4(1-B)^2\right)^{\frac{1}{2}}+\log\left[\left(\frac{C_p}{\gamma^2}+4(1-B)^2\right)^{\frac{1}{2}}-1\right]-A-\log\tfrac{4}{3}(1-B). \quad (3.13)$$

Both approximate equations, (3.9) and (3.12), indicate that $\log(\alpha\nu/\tau_0^{\frac{3}{2}})$ decreases nearly linearly with $C_p^{\frac{1}{2}}/\gamma$ for a constant stress ratio, but in the medium stress approximation $\alpha\nu/\tau_0^{\frac{3}{2}}$ increases with $(1-t)$ for constant C_p/γ^2 while it decreases in the small stress approximation (3.12). For consistency of the two approximations, $\alpha\nu/\tau_0^{\frac{3}{2}}$ should reach a maximum with respect to variation of t near $t=0{\cdot}2$, and this is confirmed by figure 5 which shows the variation of C_p/γ^2 with t for several values of $\log(\alpha\nu/\tau_0^{\frac{3}{2}})$, computed from the two approximate equations. The curves, which have been drawn to give a smooth transition from the exact values at $t=0$ to the medium-stress approximation, have maxima near $t=0{\cdot}2$ and, since C_p and $(1-t)$ both increase steadily as the layer develops, it follows that $\gamma^2(d/dC_p)\log(\alpha\nu/\tau_0^{\frac{3}{2}})$ is not only positive near zero stress but is larger than the value for continuously zero stress. In other words, the stress gradient just upstream of the position of zero stress is more than enough to satisfy the zero-stress criterion (3.13) and, as the stress gradient so defined becomes extremely large for small values of C_p/γ^2, the zero-stress criterion is also satisfied for a value of C_p/γ^2 smaller than that at zero stress. At this position, the wall stress is *not* zero but perhaps one-fifth of the initial stress ($t=0{\cdot}3$–$0{\cdot}45$). Figure 6 shows the relation between $\log(\alpha\nu/\tau_0^{\frac{3}{2}})$ and $C_p\gamma^{-2}$ for the condition of zero stress and for the condition of maximum $C_p\gamma^{-2}$ in a fixed stress gradient (nearly for $t=0{\cdot}2$). If the variation of stress gradient with pressure rise is plotted on this diagram, the resulting curve intersects the zero-stress curve from below as the pressure

† A condition for the validity of the velocity distribution (2·6) is that a region of fully turbulent flow and nearly constant stress exists, i.e. that $\tau_1^{\frac{3}{2}}/(\alpha\nu) > 20$. At ordinary Reynolds numbers, this is possible only if t exceeds 0·1, but fortunately equation (3.13) for zero wall stress is accurate and equation (3.12) is a qualitative guide to layer behaviour for t less than 0·1.

increases, touches the curve of maximum $C_p \gamma^{-2}$ and then intersects again the zero-stress curve but this time from above. Only the second intersection implies zero wall stress and presumably separation of the boundary layer.

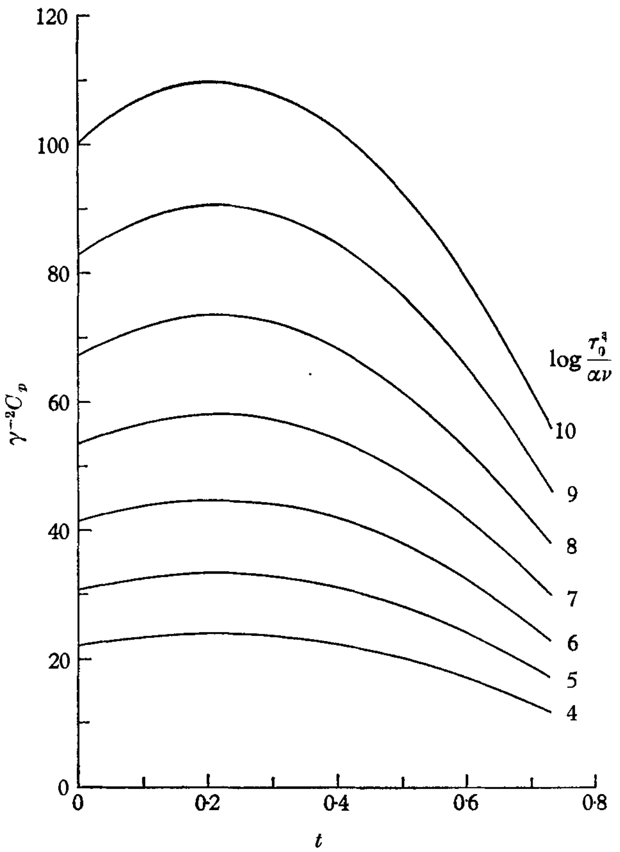

FIGURE 5. Calculated dependence of pressure rise on friction ratio and stress gradient at the wall.

4. The relation between stress gradient and pressure gradient

The adverse pressure gradient, which does not appear in the previous analysis, is related to the stress gradient and the flow acceleration by the equation of mean motion,

$$\frac{DU}{Dt} = U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{dP}{dx} + \frac{\partial \tau}{\partial y}, \tag{4.1}$$

and it is tempting to argue that the two gradients must be nearly equal within the equilibrium layer because the stream velocity is small close to the wall. This argument, though certainly valid sufficiently close to the wall, is irrelevant for the present purpose. The 'constant' stress gradient, α, appears essentially as an average value over the whole of the equilibrium layer, and the assumption that flow accelerations are negligible within the whole layer does not survive either theoretical or experimental examination. For example, the information given in figure 1 is sufficient to compute the accelerations along the streamlines. At $x = 22{\cdot}5$ ft., the streamline $\psi U_0^{-1} = 0{\cdot}05$ in. lies deep in the equilibrium layer and the acceleration along it is nearly $-0{\cdot}018U_0^2$ ft.$^{-1}$. At the same position, the pressure gradient is $0{\cdot}036U_0^2$ ft.$^{-1}$, twice as much.

A more general demonstration of the inequality of the pressure and stress gradients depends on computing from the foregoing theory the flow acceleration at the outer edge of the equilibrium layer. Starting with the velocity distribution (2.6), it may be shown that the flow acceleration there is

$$\left(U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}\right)_s = \frac{\tau_0}{K^2}\left[a\frac{dt}{dx}+b\frac{dX}{dx}\right], \tag{4.2}$$

where
$$X = \log\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu},$$

$$\begin{aligned} a(t,X) = t\bigg(&\log\frac{4t^3(1-t)}{1+t}+X+A+2\left(\frac{1}{t}-1\right)(1-B)\bigg)\\ &\times\left(\log\frac{4t^3(1-t)}{1+t}+X+A+3-\frac{2t}{1-t^2}-2(1-B)\right)-(1-B(1-t^2))\\ &\times\left[(1-3t^2)\left(\log\frac{4t^3(1-t)}{1+t}+X+A\right)+(3-2t-3t^2)\right.\\ &\left.+2(1-B)(4t^2-2t-1)-4B(1-\tfrac{3}{2}t^2)\right] \end{aligned}$$

and
$$\begin{aligned} b(t,X) = t^2\bigg(&\log\left(4t^3\frac{1-t}{1+t}\right)+X+A+2\left(\frac{1}{t}-1\right)(1-B)\bigg)\\ &-(1-B(1-t^2))\left[t(1-t^2)\left(\log\frac{4t^3(1-t)}{1+t}+X+A+1\right)\right.\\ &\left.+\tfrac{2}{3}(1-B)(2-3t-3t^2+4t^3)-2Bt^2(1-t)\right]. \end{aligned}$$

Using this acceleration to compute the stress gradient at the edge of the layer and identifying this gradient with α,

$$\alpha = \frac{dP}{dx}+\frac{\tau_0}{K^2}\left(a\frac{dt}{dx}+b\frac{dX}{dx}\right) = \frac{\tau_0^{\frac{3}{2}}}{\nu}e^{-X}. \tag{4.3}$$

Taken with the solution of the basic equations (3.4) and (3.5), which is of the form

$$C_p\gamma^{-2} = \text{fn.}\,(X,t), \tag{4.4}$$

we have a first-order differential equation for X which could be solved for any particular pressure distribution. Some characteristics of the solutions can be established for the intermediate stages of development by differentiating the approximate solution (3.9) with respect to x. Then

$$\frac{dP}{dx} = \frac{\tau_0}{K^2}\left(c\frac{dt}{dx}+d\frac{dX}{dx}\right), \tag{4.5}$$

where

$$c(t,X) = (X+A+\log t(1-t^2)+F(t)\,X^{-1})\left(\frac{1-3t^2}{1-t^2}+F'(t)\,X^{-1}\right)$$
$$-\left\{t\left(X+A+\log\frac{4t^3(1-t)}{1+t}\right)+2(1-t)\,(1-B)\right\}$$
$$\times\left(X+A+\log\frac{4t^3(1-t)}{1+t}+3-\frac{2t}{1-t^2}-2(1-B)\right),$$
$$d(t,X) = (X+A+\log t(1-t^2)+F(t)\,X^{-1})\,(1-F(t)\,X^{-2})$$
$$-t^2\left(X+A+\log 4t^3\frac{1-t}{1+t}+2\left(\frac{1}{t}-1\right)(1-B)\right).$$

Defining $q = \alpha/(dP/dx)$ and eliminating dP/dx from equations (4.3) and (4.5), we find that

$$\frac{dX}{dt} = \frac{(1-q)\,c-a}{b-(1-q)\,d}. \tag{4.6}$$

For large values of $[X+A+\log\{4t^3(1-t)\}/(1+t)]$ and not too small values of t, the coefficients are nearly

$$\left.\begin{aligned} a(t,X) &= t\left(X+A+\log\frac{4t^3(1-t)}{1+t}\right)^2,\\ b(t,X) &= t\{t-(1-t^2)+B(1-t^2)^2\}\left(X+A+\log\frac{4t^3(1-t)}{1+t}\right),\\ c(t,X) &= -t\left(X+A+\log\frac{4t^3(1-t)}{1+t}\right),\\ d(t,X) &= (1-t^2)\left(X+A+\log\frac{4t^3(1-t)}{1+t}\right) \end{aligned}\right\} \tag{4.7}$$

and so

$$\frac{dX}{dt} = \frac{-qt\left(X+A+\log\dfrac{4t^3(1-t)}{1+t}\right)}{t^2-t(1-t^2)+Bt(1-t^2)^2-(1-q)\,(1-t^2)}. \tag{4.8}$$

Since t and q are positive and less than one, dX/dt is positive if t is less than 0·57, and may be positive for $0{\cdot}57 < t < 0{\cdot}80$ if q is small enough. Positive values of dX/dt mean positive values of $d\alpha/dx$ and so stress gradient increases during the intermediate part of the layer development.

If t is less than 0·2, the approximations set out above for the coefficients become inaccurate. For small values of t and large values of $(X+A+\log 4t^3)$ the coefficients are nearly

$$\left.\begin{aligned} a(t,X) &= -(X+A+\log 4t^3)\,[1-B-t(X+A+\log 4t^3)],\\ b(t,X) &= -(1-B)\,[\tfrac{4}{3}(1-B)+t(X+A+\log 4t^3)],\\ c(t,X) &\approx \tfrac{1}{2}(X+A+\log 4t^3),\\ d(t,X) &\approx (X+A+\log 4t^3). \end{aligned}\right\} \tag{4.9}$$

The expressions for $c(t, X)$ and $d(t, X)$ have been inferred from figure 5 and their only important features are the signs and the orders of magnitude. Using these in equation (4.6),

$$\frac{dX}{dt} = -\frac{[\frac{1}{2}(1-q)+1-B-t(X+A+\log 4t^3)]\,(X+A+\log 4t^3)}{\frac{4}{3}(1-B)^2+(X+A+\log 4t^3)\,(t(1-B)+1-q)} \tag{4.10}$$

and becomes negative for small values of t.

These arguments show that a layer satisfying the requirements of the theory develops with increasing stress gradient while the wall stress is comparable with its initial value and with decreasing gradient when the wall stress is small. This behaviour is fully consistent with the hypothesis that the two gradients of Reynolds stress and of pressure approach equality at the position of zero wall stress, i.e. $q = 1$, and this is supported by the measurements of Schubauer & Klebanoff. Substitution of plausible values in equation (4.8) shows that the increase of stress gradient during the development is substantial, whatever the variation of pressure gradient, and in general the two gradients are neither equal nor proportional.

If the stress gradient and the pressure gradient do become equal at separation, the zero-stress criterion (3.13) can be put in terms of the more easily measurable pressure gradient,

$$\log\left(\frac{\nu}{\tau_0^{\frac{3}{2}}}\frac{dP}{dx}\right) = -\left(\frac{C_p}{\gamma^2}+4(1-B)^2\right)^{\frac{1}{2}} - \log\left[\left(\frac{C_p}{\gamma^2}+4(1-B)^2\right)^{\frac{1}{2}}-1\right] + A + \log\left[\tfrac{4}{3}(1-B)\right]. \tag{4.11}$$

To compare this prediction with the observations of Schubauer & Klebanoff (1951), figure 6 shows a full line representing this criterion and points representing the experimental values of C_p and dP/dx. There are two intersections and the one with the larger pressure rise corresponds almost exactly with the observed position of flow separation at $x = 25{\cdot}7 - 25{\cdot}8$ ft. The agreement depends weakly on the values chosen for the absolute constants, A, B, K, but more critically on the initial friction parameter, $\gamma = \tau_0^{\frac{1}{2}}/KU_0$. The value used, 0·083, is the value appropriate to the effective, flat-plate Reynolds number at the pressure minimum, $14{\cdot}3 \times 10^6$, and agrees with the measured slope of a semi-logarithmic plot of the velocity profile at the minimum and the relation

$$\frac{d(U/U_0)}{d(\log y)} = \frac{\tau_0^{\frac{1}{2}}}{Ky} \equiv \gamma. \tag{4.12}$$

The broken line in figure 6 represents the largest stress gradients possible for a given $C_p\gamma^{-2}$ during development to separation, and it will be noticed that some of the points lie above this line. Since the flow accelerations are always negative, stress gradients are always less than pressure gradients, and this behaviour is consistent with the theory if the stress gradients in this region are one-third less than the pressure gradients. If, consistently with the proposition that stress gradient increases during medium-stress development, the gradient ratio q is about 0·5 for somewhat smaller values of $C_p\gamma^{-2}$, the first intersection of the stress-gradient curve ($\log(\alpha\nu/\tau_0^{\frac{3}{2}})$ vs. $C_p\gamma^{-2}$) with the critical curve is at

$$x = 23{\cdot}5\,\text{ft.},\quad \log\frac{\alpha\nu}{\tau_0^{\frac{3}{2}}} = -7{\cdot}5,\quad C_p\gamma^{-2} = 60,\quad C_p = 0{\cdot}38,$$

equivalent to $t = 0{\cdot}43$. The wall stress calculated from the observed velocity distribution at $x = 23{\cdot}5$ ft. gives $t = 0{\cdot}40$, in satisfactory agreement.

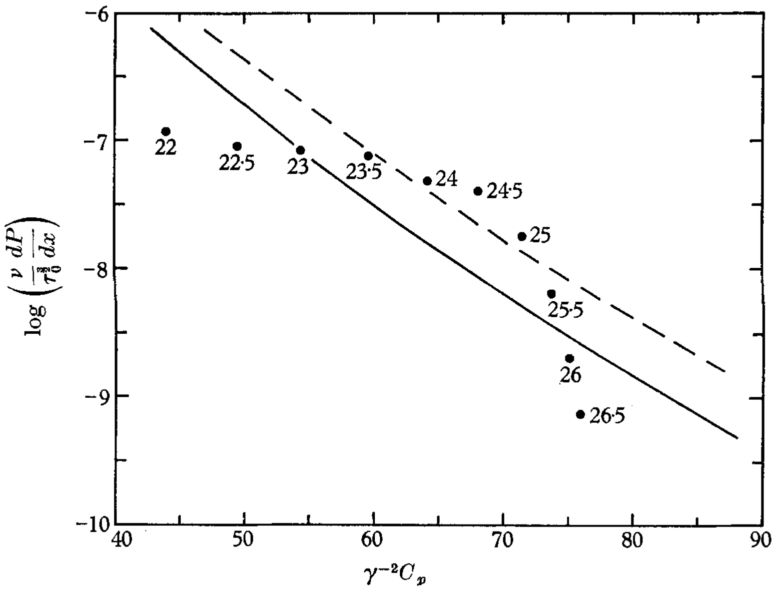

FIGURE 6. Comparison of zero-stress prediction (equation (4.11)) with measurements of Schubauer & Klebanoff (1951). The numbers alongside the experimental points are the downstream positions in feet, the full line is the zero-stress condition and the broken line gives the maximum possible values of $\log(\alpha\nu/\tau_0^{\frac{3}{2}})$.

The second intersection is at

$$x = 25{\cdot}8\,\text{ft.},\quad \log\frac{\alpha\nu}{\tau_0^{\frac{3}{2}}} = -8{\cdot}5,\quad C_p\gamma^{-2} = 74{\cdot}5,\quad C_p = 0{\cdot}51$$

and corresponds closely with the observed position of separation.

5. The pressure distribution near separation

The development of the boundary layer has been discussed so far with the implicit assumption that the pressure distribution is predetermined and forms part of the specification of the flow, but the necessity for a rapid decrease in pressure gradient as the position of zero stress is approached means that the pressure distribution in this region is not completely arbitrary. Measurements in separating flows show that the total decrease in gradient before separation is large, so that the statement, 'Separation takes place when the pressure gradient has fallen to one-fifth of its average value', could replace the theoretical criterion without appreciable error. This suggests that the pressure distribution near the point of separation may be characteristic of the phenomenon of separation and that inclusion of it in the specification of the flow is equivalent to being told the position of separation. If the prediction of separation is to be more than a demonstration of the self-consistency of the theoretical assumptions, details of the pressure distribution should not form part of the initial data. Fortunately, the rapid weakening of the gradient is confined to a comparatively small region around the separation point, where the rate of increase of boundary-layer thickness is large. For example, in the layer studied by Schubauer & Klebanoff, the pressure gradient is nearly constant from just past the pressure minimum at $x_0 = 17{\cdot}5$ ft. to $x = 23{\cdot}5$–$24{\cdot}0$ ft. ($C_p = 0{\cdot}41$–$0{\cdot}44$), and then the gradient decreases rapidly with separation occurring at $x = 25{\cdot}7$–$25{\cdot}8$ ft. ($C_p = 0{\cdot}51$). The small range permits a useful distinction between the effects on the pressure distribution of the flow separation and of the layer thickening.

The speculation that the pressure distribution near the separation point has a characteristic form which is independent of the particular flow system can be supported by consideration of the relation between the pressure distribution in the real flow and in an inviscid flow with the same boundaries and a free streamline leaving the surface at the separation point. In the real flow the pressure near the separated layer soon becomes constant, since it encloses stagnant fluid of negligible kinetic energy, and this pressure is the pressure on the free streamline of the inviscid flow.† The surface distribution of pressure in the inviscid flow will be very nearly the same as the distribution in the real flow well upstream of separation, where the layer thickness is small and also well downstream where the fluid is stagnant, but not near the separation point. There the rapid increase of boundary-layer thickness displaces streamlines in the effectively inviscid flow outside the layer and may cause appreciable pressure differences across the layer itself. The changes in wall pressure depend on additional curvature of streamlines consequent on growth of the boundary layer and, since the growth is determined by the real pressure distribution and by the parameter describing the initial layer, the real pressure distribution is a function of the inviscid distribution and these parameters. But the differences are appreciable only in the immediate neighbourhood of the separation point and so depend only on flow conditions near this point. In this restricted region, the inviscid pressure distribution is approximately one with constant pressure gradient, $(dP/dx)_i$, to the separation point and zero gradient beyond it. In the non-dimensional form required by the preceding theory of layer development, this means that

$$\frac{\nu}{\tau_0^{\frac{3}{2}}}\frac{dP}{dx} = \text{fn.}\left[\frac{\nu}{\tau_0^{\frac{3}{2}}}\left(\frac{dP}{dx}\right)_i,\quad \gamma^{-2}C_p, \gamma^{-2}C_{p,s}\right], \tag{5.1}$$

where C_p rather than x is used as the independent variable, and $C_{p,s}$, the pressure coefficient at separation, specifies the pressure on the free streamline. To the approximation that $(dP/dx)_i$ may be regarded as constant, the real pressure

† Notice that the inviscid flow is determined by the flow boundaries, the position of separation and the pressure on the free streamline.

gradient approaches this value upstream of the separation point and zero downstream. Combining equation (5.1) with the zero-stress criterion, which expresses pressure coefficient at separation as a function of pressure gradient at separation, it follows that

$$\gamma^{-2}C_{p,s} = F\left[\frac{\nu}{\tau_0^{\frac{3}{2}}}\left(\frac{dP}{dx}\right)_i\right] \tag{5.2}$$

and that

$$\frac{\nu}{\tau_0^{\frac{3}{2}}}\frac{dP}{dx} = f[\gamma^{-2}C_p, \gamma^{-2}C_{p,s}]. \tag{5.3}$$

These arguments depend on the experimental observation that layer thickening exerts an appreciable effect on the pressure gradient only in the neighbourhood of separation and this phenomenon probably depends on the close connexion between pressure gradient and displacement thickness for small values of the stress ratio. For these small values, the contribution of the inner layer to the displacement thickness is

$$\delta_i^* = \frac{(U_1-U_t)\,y_s - (4(1-B)/3K)\,\alpha^{\frac{1}{2}}y_s^{\frac{3}{2}}}{U_1} = \frac{\tau_0}{\alpha}\left[\frac{U_1-U_t}{U_1} - \frac{4(1-B)}{3}\gamma\right], \tag{5.4}$$

where

$$U_t = \frac{\tau_1^{\frac{1}{2}}}{K}\left[\log\frac{4\tau_1^{\frac{3}{2}}}{\alpha\nu} + A - 2(1-B)\right].$$

The contribution from the outer layer varies slowly with position and does not depend directly on the stress gradient, α. From (5.4) it follows that a decrease in pressure gradient, which is an upper bound to the stress gradient, causes a corresponding increase in displacement thickness whose effect is to make the pressure gradient still smaller. Once this contribution to the thickness is large enough to change the pressure gradient appreciably, the layer is likely to become unstable with respect to downstream development and to thicken at a rate not controlled by outside influences. It is also likely that this runaway development begins at some critical value of the stress ratio and that a layer allowed to develop past this stage must separate.† Indeed, equation (5.4) shows that, at this stage of development, weakening of the gradient by external action merely accelerates the layer thickening.

If these conclusions are correct, the criterion for separation may be expressed in the form (5.2) as a dependence of pressure rise to separation on the inviscid pressure gradient just upstream of the separation point. The dependence could be established by measuring pressure distributions in separating flows and from them inferring the characteristic pressure distributions implied by equation (5.3). In figure 7, one such distribution is shown, calculated from the measurements of Schubauer & Klebanoff, with some conjectural curves for other values of $\nu/\tau_0^{\frac{3}{2}}(dP/dx)_i$. Evidently the dependence of $\nu/\tau_0^{\frac{3}{2}}(dP/dx)_i$ on $\gamma^{-2}C_{p,s}$ must be similar to that of $\nu/\tau_0^{\frac{3}{2}}(dP/dx)_s$, and a possibility is that $(dP/dx)_s$ is a constant fraction of $(dP/dx)_i$. A curve has been drawn using the fraction,

$$\left(\frac{dP}{dx}\right)_s \bigg/ \left(\frac{dP}{dx}\right)_i = 0{\cdot}24, \tag{5.5}$$

† Flow with continuously zero wall stress is a special case, $dt/dx \equiv 0$, and these arguments are not inconsistent with its existence (Stratford 1959). They do imply transient separation before the zero-stress flow is established.

to agree with the measurements of Schubauer & Klebanoff at

$$\log\left[\frac{\nu}{\tau_0^{\frac{3}{2}}}\left(\frac{dP}{dx}\right)_i\right] = -7.$$

From a curve of this kind, a condition for separation is defined in terms of pressure distribution in an inviscid flow.

Although the effects of weakening of the pressure gradient can be described in this comparatively simple way, the problem of predicting separating flow in a system with specified boundaries is still difficult except in special circumstances. In general, inviscid flows are possible with various points of separation of the

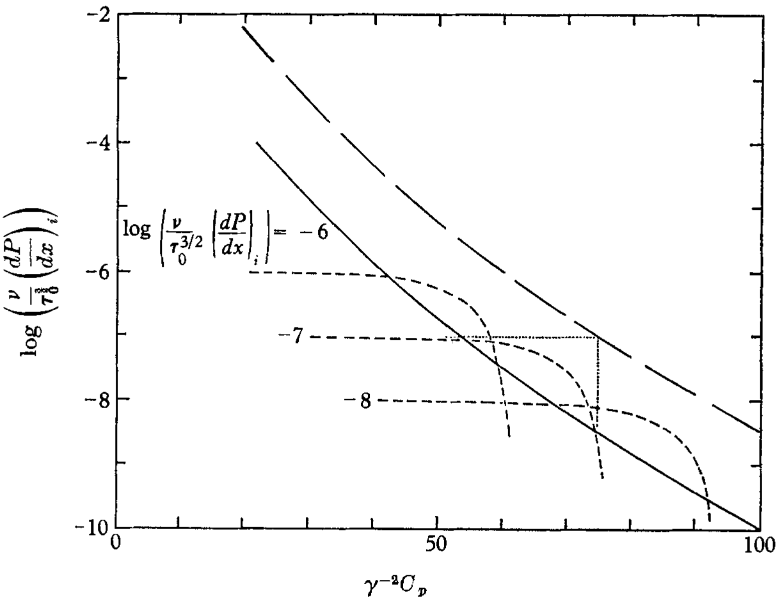

FIGURE 7. Characteristic pressure distributions in separating boundary layers. The curve for $\log[\nu/\tau_0^{\frac{3}{2}}(dP/dx)_i] = -7$ is taken from the measurements of Schubauer & Klebanoff (1951) but the other two are conjectural. The full line represents the zero-stress condition (4.11) and the broken line the suggested relation between $C_{p,s}$ and $(dP/dx)_i$ (equation (6.2)).

free streamline and various pressures on this streamline, and the problem consists of the determination of these two parameters. Since $C_{p,s}$ and $(dP/dx)_i$ are functions of these parameters, the criterion for separation (5.2) imposes a relation between them, but to decide which position of separation (and corresponding stagnant pressure) is the real one requires a knowledge of the behaviour of the separated layer downstream of the separation. In some flows, the separated layer may extend far downstream and then the pressure in the stagnant region may be the pressure at infinity, but usually the layer re-attaches itself either to the surface or, as in flows past cylinders, to another separated layer, forming a closed stagnant region. The work of Bourque (1959) and of Sawyer (1960) on the re-attachment of a jet issuing parallel to a flat plate suggests that the appropriate inviscid flow is one with a re-entrant free streamline and that the strength of the sink necessary for the re-entry is the total rate of entrainment by the turbulent mixing layer which in the real flow occupies the position of the free streamline. The imposition of a condition of this kind would provide another relation between position of separation and stagnant pressure, and a single flow would be defined.

Even if the conditions for closure of the stagnant region were established, the prediction of separating flow around a bluff body (e.g. a circular cylinder) would be far from easy but most of the difficulty disappears if the width of the region of separated flow is small compared with the extent of the region of pressure rise. This occurs in flow separating at the rear of an aerofoil or in a diffuser of small angle. In such flows, the inviscid pressure distribution up to the point of separation depends little on the position of separation and the inviscid gradient $(dP/dx)_i$ may be approximated by the value in the non-separating flow without serious error.

6. Discussion

The development of this theory may have given the impression that its assumptions are necessarily valid if the pressure gradient is large compared with the initial stress gradients, but by itself this condition ensures only that the outer layer has the required property of conserving total-head and Reynolds stress along streamlines. The conditions for an inner layer with the assumed structure are more complex and impose restrictions on the form of the pressure distribution as well as on the magnitude of the gradients. We may recall that the inner layer is the part of the flow in which the distributions of total-head and Reynolds stress have been strongly modified through increased production and dissipation of turbulent energy, and it is assumed that the product of this modification is a linear stress, equilibrium layer. However, the conditions for existence of an equilibrium layer are not necessarily satisfied over the whole inner layer as can be shown by considering the variation with x of the calculated stream function at the junction of the two layers. It is

$$\psi_s = t(1-t^2)\frac{\tau_0^{\frac{3}{2}}}{\alpha K}\left[\log\left(\frac{4t^3(1-t)}{1+t}\frac{\tau_0^{\frac{3}{2}}}{\alpha\nu}\right)+A+\frac{2}{3}\frac{2-t-4t^2}{t(1+t)}(1-B)-\frac{2t}{1+t}B\right] \quad (6.1)$$

and, if $d\psi_s/dx < 0$, streamlines through the junction points are emerging from the inner layer and total-head and Reynolds stress on them will not be the same as at the pressure minimum. In this event, the inner layer of modified flow ends further from the wall than the calculated distance, $y_s = \tau_0/\alpha$, and the equilibrium layer forms only a part of it. In practice, shallow excursions of a streamline into the inner layer will not change stress and total-head by significant amounts and the identification of the inner and equilibrium layers is still possible if negative values of $d\psi_s/dx$ are small compared with $\psi_s/(x_s-x_0)$. It is interesting that this condition can be met only if the stress gradient decreases rapidly as the wall stress approaches zero and that this decrease is a condition for validity of the basic assumptions as well as a consequence of them.

The more usual methods of predicting layer development in adverse pressure gradients depend on the equation for the momentum integral and on procedures for relating the velocity distributions to the wall stress and a shape-parameter. The Stratford approach postulates a particular stress distribution, linear in the inner layer and 'frozen' in the outer layer, and deduces a velocity distribution consistent with it, thus satisfying automatically the momentum-integral equation. One advantage of this approach is that the boundary-layer approximation need be valid only within the inner layer, where flow velocities and curvatures are small and where pressure is less likely to change with distance from the wall. Another advantage, pointed out by Stratford (1959), is that the influence of Reynolds number and thickness of the initial layer on separation is easily found. In terms of the 'inviscid' pressure gradient, the criterion for zero stress is

$$\log\left[\frac{\nu}{\tau_0^{\frac{3}{2}}}\left(\frac{dP}{dx}\right)_i\right] = -\left(\frac{C_{p,s}}{\gamma^2}+4(1-B)^2\right)^{\frac{1}{2}} - \log\left[\left(\frac{C_{p,s}}{\gamma^2}+4(1-B)^2\right)-1\right]+1{\cdot}51+A \tag{6.2}$$

obtained from equation (4.11) by inserting the speculative relation (5.4). At constant Reynolds number, the left-hand term varies slowly with the friction parameter γ, related to layer thickness by $U_0\delta_0/\nu = 1/K\gamma\, e^{\gamma^{-1}-A}$, and so $C_{p,s}$ is nearly proportional to γ^2, i.e. to the initial coefficient of skin friction. Next, defining x_0 as the distance from the leading-edge of a flat plate in a stream of velocity U_0 at which the skin friction has the value at the pressure minimum and using the relation (Townsend 1956)

$$\frac{U_0 x_0}{\nu} = K^{-3} I_1 \gamma^{-2} \exp\left[\gamma^{-1} - A\right],$$

the criterion becomes

$$(C_{p,s}+4(1-B)^2\gamma^2)^{\frac{1}{2}} = 1-\gamma\left[\log\left[\frac{x_0}{U_0^2}\left(\frac{dP}{dx}\right)_i\right] + \log\left[(C_{p,s}+4(1-B)^2\gamma^2)^{\frac{1}{2}}-\gamma\right]-0{\cdot}86\right]. \tag{6.3}$$

Unless the position of transition changes considerably, x_0 should not vary appreciably with Reynolds number of the flow, and, to a fair approximation in any particular flow, the difference from unity of the pressure rise coefficient at separation is proportional to γ, i.e. to the square root of the initial friction coefficient.

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